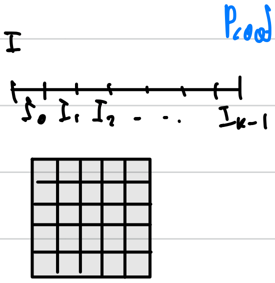
Math 451: Introduction to General Topology
Lecture 23

Examples. (a) For $1 \leq d < \infty$, every bdd set in $\mathbb{R}^d$ is totally bdd.



Proof. If a set is bdd then it is contained in a large box $B := (-n, n)^d$, so it suffices to show that each box of the form I^d, where $I \subseteq \mathbb{R}$ is an open interval, is totally bdd. But for each $\varepsilon > 0$, taking $k > \frac{\sqrt{d}\,\mathrm{diam}(I)}{\varepsilon}$, we can split I into k equal intervals, each having $\mathrm{diam} = \mathrm{diam}(I)/k$: $I = \bigcup_{i<k} I_i$. Then

$$B = \bigcup_{(i_1,\dots,i_d) \in k^d} I_{i_1} \times I_{i_2} \times \dots \times I_{i_d}$$

and each box $I_{i_1} \times I_{i_2} \times \dots \times I_{i_d}$ has side length $\delta := \mathrm{diam}(I)/k$, so the diam B in the Euclidean distance is $= \sqrt{\underbrace{\delta^2 + \delta^2 + \dots + \delta^2}_{d}} = \sqrt{d} \cdot \delta = \sqrt{d} \cdot \mathrm{diam}(I)/k < \varepsilon$. $\square$

(b) $2^{\mathbb{N}}$ is totally bdd (any $k^{\mathbb{N}}$) but $\mathbb{N}^{\mathbb{N}}$ isn't. (HW)

For a finite collection $\mathcal{P}$ of bdd sets in a metric space, denote $\mathrm{diam}\,\mathcal{P} := \max_{P \in \mathcal{P}} \mathrm{diam}(P)$.

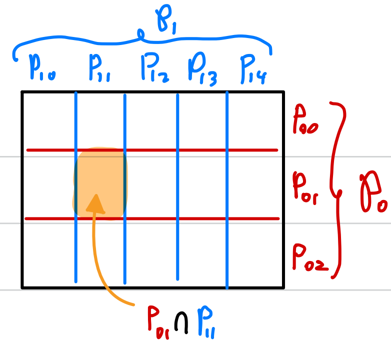
Def. A **partition** of a set X is a collection $\mathcal{P}$ of **pairwise disjoint** subsets of X whose union is X, i.e. $X = \bigcup \mathcal{P} = \bigcup_{P \in \mathcal{P}} P$. We denote this by $X = \bigsqcup \mathcal{P}$ to indicate that the union is disjoint.
Note that a partition of X is in particular a cover of X.

Disjointification. For a finite sequence $A_0, A_1, \dots, A_n$ of sets, their **disjointification** is the sequence $A_0', A_1', \dots, A_n'$ where $A_k' := A_k \setminus \bigcup_{i<k} A_i$. Clearly, the sets $A_0', \dots, A_n'$ are pairwise disjoint, $A_k' \subseteq A_k$, and $\bigcup_{k \leq n} A_k' = \bigcup_{k \leq n} A_k$.

Obs. A metric space X is totally bdd $\iff$ $\forall \varepsilon > 0$ $\exists$ finite partition $\mathcal{P}$ of X with $\mathrm{diam}\,\mathcal{P} < \varepsilon$.

Proof. $\Leftarrow$: By def., and for $\Rightarrow$, one just disjointifies finite covers. $\square$

For covers (e.g. partitions) $\mathcal{U}$ and $\mathcal{V}$ of a set X, we say that $\mathcal{U}$ **refines** $\mathcal{V}$ if each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$.



Refinement. In a metric space X, if $\mathcal{P}_0$ and $\mathcal{P}_1$ are finite partitions of X then $\mathcal{P}_0 \wedge \mathcal{P}_1 := \{P_0 \wedge P_1 : P_0 \in \mathcal{P}_0 \text{ and } P_1 \in \mathcal{P}_1\}$ is a finite partition of X refining both $\mathcal{P}_0$ and $\mathcal{P}_1$ and $\operatorname{diam}(\mathcal{P}_0 \wedge \mathcal{P}_1) \leq \min(\operatorname{diam} \mathcal{P}_0, \operatorname{diam} \mathcal{P}_1)$.

Cor. A metric space (X,d) is totally bdd $\iff$ $\exists$ sequence $(\mathcal{P}_n)_{n\geq 1}$ of finite partitions of X with $\operatorname{diam}(\mathcal{P}_n) < \frac{1}{n}$ and s.t. $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$.

Proof. $\Leftarrow$. By def. $\Rightarrow$. For each $n \geq 1$, there is a finite partition $\mathcal{P}_n$ of X with $\operatorname{diam} \mathcal{P}_n < \frac{1}{n}$. Let $\tilde{\mathcal{P}}_n := ((\mathcal{P}_1 \wedge \mathcal{P}_2)\wedge)\ldots\wedge \mathcal{P}_n)$, i.e. $\tilde{\mathcal{P}}_1 = \mathcal{P}_1$ and $\tilde{\mathcal{P}}_{n+1} := \tilde{\mathcal{P}}_n \wedge \mathcal{P}_{n+1}$ (inductively). Then $\tilde{\mathcal{P}}_{n+1}$ refines $\tilde{\mathcal{P}}_n$ and $\operatorname{diam} \tilde{\mathcal{P}}_n \leq \operatorname{diam} \mathcal{P}_n < \frac{1}{n}$. $\square$

Def. A metric space has the **Heine–Borel property** if it is complete and totally bdd.

The Heine-Borel Theorem. For a metric space X, TFAE:

(1) X is compact, i.e. every open cover has a finite subcover.

(2) X is sequentially compact, i.e. every sequence has a convergent subsequence.

(3) X has the Heine-Borel property, i.e. X is complete and totally bdd.

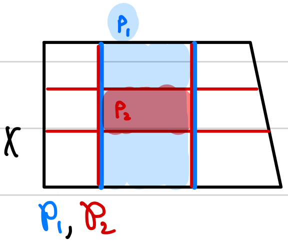
Proof. $(1) \Rightarrow (2)$. In HW you're asked to prove that this implication holds for all 1^{st} ctbl spaces more generally.

$(2) \Rightarrow (3)$. For completeness, take a Cauchy sequence, so by (2) it has a convergent subsequence, hence the whole sequence converges by the Cauchy property.

For total bddness, suppose X is not totally bdd, so $\exists\, \varepsilon > 0$ s.t. $\nexists$ finite cover of X with sets of diam $< \varepsilon$. Let $\delta := \varepsilon/4$. We build a sequence (x_n) of points of pairwise distance $\geq \delta$, as follows: let $x_0 \in X$ be any point, and assuming $\{x_0, \ldots, x_{n-1}\}$ are defined, consider $\mathcal{P}_n := \{B_\delta(x_0), \ldots, B_\delta(x_{n-1})\}$. This is a finite collection of sets of diam $2\delta = \varepsilon/2$, hence $\mathcal{P}_n$ is not a cover of X, so pick $x_n \in X \setminus \bigcup \mathcal{P}_n = X \setminus \bigcup_{i<n} B_\delta(x_i)$, so $d(x_n, x_i) \geq \delta$ $\forall i < n$. This finishes the construction of the sequence (x_n).

Now (x_n) has no Cauchy subsequence, hence no convergent subsequence.

(3)$\Rightarrow$(1). Let $\mathcal{U}$ be an open cover of X and suppose towards a contradiction that it has no finite subcover. Total boundedness of X gives us a sequence $(\mathcal{P}_n)$ of finite partitions of X with $\operatorname{diam} \mathcal{P}_n < \frac{1}{n}$ and such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. We now do the same argument as in the proof that $2^{\mathbb{N}}$ is compact, using these $\mathcal{P}_n$ as the "levels of the tree". Namely: since X doesn't have a finite subcover in $\mathcal{U}$ and $\mathcal{P}_1$ is finite, then $\exists\, P_1 \in \mathcal{P}_1$ which can't be covered by a finite subset of $\mathcal{U}$. Since P_1 is a union of finitely many sets in $\mathcal{P}_2$, one of these sets P_2 also doesn't have a finite subcover in $\mathcal{U}$. Thus, $P_2 \in \mathcal{P}_2$ and $P_2 \subseteq P_1$. Continuing this way (by induction), we obtain a decreasing sequence $(P_n)_{n \geq 1}$ s.t. $P_n \in \mathcal{P}_n$, hence $\operatorname{diam} P_n < \frac{1}{n}$.



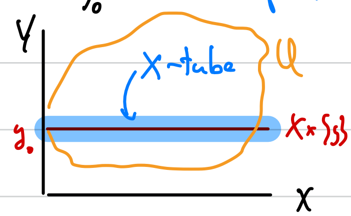
Then $(\overline{P}_n)_{n \geq 1}$ is a decreasing sequence of closed sets of vanishing diameter since $\lim_{n\to\infty} \operatorname{diam}(\overline{P}_n) = \lim_{n\to\infty} \operatorname{diam}(P_n) \leq \lim_{n\to\infty} \frac{1}{n} = 0$. By completeness, $\exists\, x \in \bigcap_{n \geq 1} \overline{P}_n$. Since $\mathcal{U}$ is a cover of X, $\exists\, U \in \mathcal{U}$ with $U \ni x$. Hence $B_{\frac{2}{n}}(x) \subseteq U$ for some $n \geq 1$. Since $x \in \overline{P}_n$ and $\operatorname{diam} P_n < \frac{1}{n}$, we have $P_n \subseteq B_{\frac{2}{n}}(x) \subseteq U$ so $\{U\} \subseteq \mathcal{U}$ is a finite cover of P_n, contrary to its choice. $\square$

<u>Remark.</u> In general, if a top. space is metrizable, it might have compatible metrics which are complete and some which are not complete. Example. On $(-1,1)$, the usual metric is incomplete, but $(-1,1)$ is homeomorphic to $(-\infty,\infty) = \mathbb{R}$ (by $f(x) := \frac{x}{1-|x|}$?) so we can copy the complete metric from $\mathbb{R}$ to $(-1,1)$ by setting $d_f(x,y) := |f(x) - f(y)|$, where $f : (-1,1) \xrightarrow{\sim} \mathbb{R}$ is a homeomorphism. So $(-1,1)$ has at least one complete and at least one incomplete compatible metric. However, this is not the case with compact metrizable spaces:

<u>Cor.</u> If a compact top space is metrizable, then **all** compatible metrics are complete.
<u>Proof.</u> This is (1)$\Rightarrow$(3) of the Heine-Borel theorem. $\square$

Compactness and products.

Let X, Y be top. spaces. An X-slice is a set of the form $X \times \{y\}$ for some $y_0 \in Y$. An open X-tube is a set of the form $X \times V$ for some nonempty open $V \subseteq Y$.



Tube lemma. Let X, Y be top spaces with X being compact. Then for each open $W \subseteq X \times Y$, if W contains an X-slice $X \times \{y_0\}$, $y_0 \in Y$, then W contains an open X-tube $X \times V_0 \supseteq X \times \{y_0\}$.

Proof. For each $x \in X$, since $(x, y_0) \in W$ $\exists$ open rectangle $(x, y_0) \in U_x \times V_x \subseteq W$. Thus, $\{U_x\}_{x \in X}$ is an open cover of X, hence $\exists$ finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$. Take $V_0 := V_{x_1} \cap V_{x_2} \cap \dots \cap V_{x_n}$, then $\forall i = 1, \dots, n$, $U_{x_i} \times V_0 \subseteq U_{x_i} \times V_{x_i} \subseteq W$, hence

$$W \supseteq \bigcup_{i=1}^{n} U_{x_i} \times V_0 = \left(\bigcup_{i=1}^{n} U_{x_i}\right) \times V_0 = X \times V_0. \qquad \square$$